

RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

M.A./M.Sc. THIRD SEMESTER EXAMINATION, JANUARY 2015

SECOND YEAR

MATHEMATICS

Date : 19/01/2015

Time : 11 am – 1 pm

Paper : XIV [Probability Theory & Statistics]

Full Marks : 50

Answer any two questions from Q.No 1 - 4 :

[2×20]

1. Suppose that $X_1, X_2, \dots, X_n$ are i.i.d observations from $U(\theta - 1/2, \theta + 1/2), \theta \in \mathbb{R}$.
 - a) Derive the joint p.d.f of $(X_{(1)}, X_{(n)})$. Find $\text{Cov}(X_{(1)}, X_{(n)})$. [10]
 - b) Show that there is no unique maximum likelihood estimator of θ . Find the maximum likelihood estimator with the smallest mean squared error. Is the resulting estimator unbiased for θ ? [10]
2. Suppose that $Y_i \sim N(\beta(i/n), 1), 1 \leq i \leq n$, and they are independent. Consider the hypothesis $H_0: \beta = 0$ versus $H_1: \beta \neq 0$. Find the likelihood ratio test at level α . Hence, find a confidence interval for β with confidence level $1 - \alpha$. [20]
3. Suppose that we have two independent random samples : $X_1, \dots, X_n$ are exponential (θ) and $Y_1, \dots, Y_m$ are exponential (μ)
 - a) Find the LRT of $H_0: \theta = \mu$ versus $H_1: \theta \neq \mu$. [7]
 - b) Show that the test derived in (a) can be based on the statistic $T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j}$. [3]
 - c) Find the distribution of T when H_0 is true. [10]
4. Let p be the probability that a toss of a given coin results in a head. Suppose that you continue to toss the coin until the first head is observed, and define X to be the number of trials needed.
 - a) Find the p.m.f of X . [5]
 - b) Let $M \geq 1$ be a fixed integer. Find the p.m.f of $Y = \min(M, X)$. [7]
 - c) Derive the conditional p.m.f of X given Y . [8]

Answer any one questions from Q.No 5 - 7 :

[1×10]

5. Suppose $X_1, \dots, X_n$ are i.i.d Poisson (λ)
 - a) Find the best unbiased estimator of $e^{-\lambda}$, the probability that $X = 0$. [4]
 - b) Calculate the asymptotic relative efficiency of the above estimator in (a) with respect to the MLE. Which one you prefer and why? [6]
6. a) Let $U_1, U_2, \dots$ be i.i.d observations from $U(0, \theta)$. Define $Y_n = n^{-1} \sum_{i=1}^n \ell_n(U_i)$. Show that $\sqrt{n}\{Y_n + 1 - \ell_n(\theta)\}$ converges in distribution to $N(0, 1)$ as $n \rightarrow \infty$. [5]
b) Using (a) show that $e\{\prod_{i=1}^n U_i\}^{1/n}$ is a consistent estimator of estimator of θ . [5]
7. Let $X_1, X_2, \dots$ be i.i.d observations for $\text{Bin}(1, p)$ and Y be an independent $\text{Poi}(\lambda)$ variable. Let $Z = \sum_{i=1}^Y X_i$. Derive the probability generating function of Z . Hence, find the p.m.f of Z . [10]